IEOR160：Operations Research I

Midterm Exam
October 29, 2008

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Name: ____ (please print)

SID: ____

- Clearly state all the mathematical expressions that are needed to solve the problems. **No credit will be given to numerical answers without the proper setup.**
- Answer each of the following questions in the space provided. If you need more space to show major computations you performed to obtain your answer for a particular problem, use the back of the preceding page.
- You can quote and use any result stated in class or in the main body of the textbook as well as well-known general mathematical results but **no** references to other sources (including homework and textbook exercises) are allowed.
- Present your work in an organized and neat fashion.
- **Assume that all the functions are twice continuously differentiable.**
- Good luck!

Problem	1 (35)	2 (35)	3 (35)	Total (105)
Score				

A score of 100 would be considered as perfect.
Problem 1 (35 points)

A farmer wishes to fence in a rectangular pen for her animals. She only has 200 feet of fencing to use, but the back wall of the barn is 40 feet long and she can use it for part or all of one side of the pen.

a) Formulate a nonlinear programming problem so that the area of the pen will be as large as possible. Use only one variable in the formulation.
b) Write the KKT conditions for the problem.
c) Solve the system you got in (b).
d) Is the solution you got from (c) the optimal solution for the problem? **Explain.**

Solution

a) Let x be the length of the longer side of the rectangular.
   Max $x(240/2-x)$
   s.t $x \geq 40$
   $240/2-x \geq 0$
b) Optimality conditions for the problem are
   $120-2x+\lambda_1-\lambda_2=0$
   $\lambda_1(x-40)=0$
   $\lambda_2(120-x)=0$
   $\lambda_1, \lambda_2 \geq 0$
   $x \geq 40$
   $120-x \geq 0$
c) $x=60, \lambda_1=0, \lambda_2=0$
d) Since the second-order derivative of the objective function is -2, it is a concave function. And all the constraints are linear therefore convex, also the constraints qualification conditions are satisfied. KKT conditions is necessary and sufficient for optimality:
   Thus, x=40 is the optimal solution.

Problem 2 (35 points)

Consider the following problem:

(P) $\quad z = \max \quad -x^2 - y^2$

$\quad\quad$ **s.t.** $\quad (x-1)^2 - y^2 = b$

For questions (a)-(c) assume that b=0.

a) Write the Lagrangian function for (P).
b) Determine **all** the points at which the gradient of the Lagrangian function is zero.
c) Given the fact that (P) has a global solution, find it.
d) Find **$\partial z/\partial b$**.

Solution

a) $L(x,\lambda) = -x^2-y^2+\lambda(-(x-1)^2+y^2)$

b) $-2x-2\lambda(x-1)=0$

$-2y+2\lambda y=0$

$-(x-1)^2+y^2=0$

Thus, $x= ½, y=-1/2, \lambda=1$

or $x=1/2, y=1/2, \lambda=1$

c) Since (P) has a global solution, it is either a point such that constraint qualifications are not satisfied or a point that constraint qualifications and KKT conditions are satisfied.

First consider the point where constraint qualifications are not satisfied:

$(2(x-1),-2y)=(0,0) \Rightarrow x_0=1, y_0=0$. Since $z(1,0)=-1<z(1/2,1/2)=-1/4$, (x_0,y_0) cannot be an optimal solution.

Thus, the optimal solution must be a point that satisfies constraint qualification and KKT conditions.

Since both (1/2,1/2) and (1/2, -1/2) satisfy the constraints qualification, and z(1/2,1/2)=z(1/2, -1/2). Both (1/2,1/2) and (1/2, -1/2) are global optimal solutions for (P).

d) **$\partial z/\partial b|(b=0)=\lambda=$** 1.

Problem 3 (35 points)

Read each of the following statements carefully to see whether it is *true* or *false*. **Justify your answers (no credit for answers without justification!)**

For the following question suppose **x** is an n-dimensional column vector and let ∇**f(x)** denote the gradient of a function **f** at **x** (presented as a row vector).

a) Suppose ∇**f(x*)d=0** for all n-dimensional column vectors d. Then **x*** is a local maximum point of **f**.

   False. For example, $f(x)=x^2$, x*=0. $\nabla f(x^*)=0$, thus $\nabla f(x^*)d=0$. But x* is a local minimum instead of a maximum.

b) If ∇**f(x*)=0** and all leading principal minors of the Hessian of **f** are negative for all the points in $\mathbf{R}^n$, then **x*** is a global maximum point for **f**.
   Note – there are no constraints in this question.

   False. For example, $f(x,y)=-x^2+y^2$, (x*,y*)=(0,0). H(x*,y*) has leading principal minors all negative, but (x*,y*) is not a global maximum point.

c) Consider:

   (P1) **Max** **f(x)**
   s.t. $g_i(x) \geq b_i$, **i=1,…m**

   Assume **f** and g_i (i=1,…,m) are all concave functions. If **x*** satisfies the KKT conditions and the constraint qualifications, then **x*** is a global maximum point.

   True. Since g_i concave, $-g_i$ convex. Therefore, we maximize a concave function f under convex region, therefore, if constraint qualifications satisfy, KKT condition is sufficient for optimality.

d) Consider:

   (P2) **Max** **f(x)**
   s.t. **ax≤b** (where **a** is an n-dimensional row vector)

   Assume **f** is a concave function. If **x*** satisfies **ax*<b** and ∇**f(x*)=0**, then **x*** is a global maximum point.

   True. The constraint qualification is satisfied as x* since x* is not binding at any constraint. Let $\lambda^*=0$, then (x^*,λ^*) is a solution for the KKT condition of (P2). Since f is a concave function and constraints are linear therefore convex, the KKT condition is sufficient for optimality. Thus, x* is a global maximum.

e) Consider:

(P3) **Max** **f(x)**

s.t. $\mathbf{g_i(x) \leq b_i, i=1,\ldots m}$

Assume **f** is concave and $\mathbf{g_i}$ (i=1,…,m) are convex. Let **x*, y*** be two (global) optimal solutions for (P3). Then, **0.5x*+0.5y*** is also a (global) optimal solution for (P3).

True. Since g_i are convex, for any i, we have $g_i(0.5x^*+0.5y^*) \leq 0.5g_i(x^*)+0.5g_i(y^*) \leq b_i$, therefore, 0.5x*+0.5y* is feasible.

Since both x* and y* are optimal solutions, f(x*)=f(y*). Also since f is concave, $f(0.5x^*+0.5y^*) \geq 0.5f(x^*)+0.5f(y^*)=f(x^*)$, therefore 0.5x*+0.5y* is also an optimal solution for (P3).