IEOR 160, Homework #5 Solution

1.

A concave function cannot have a minimizer over an equality constrained feasible region.

False. Consider function $f(x) = \ln x$, over the feasible region with equality constraint $x = 1$. Then $x = 1$ is a minimizer.

If the objective function of an optimization problem is convex and the feasible region is convex, then it is a minimization problem.

False. Whether a problem is a maximization problem or minimization problem is nothing related to the property of objective function and feasible region.

If f is continuously differentiable function, then all its local maxima is among its stationary points.

True.

If x is a local maximum of a concave function, then there exists a direction vector d for which the directional derivative at x is negative.

False. $f(x) = 1$ is a concave function. And point $x = 1$ is a local maximum. But for any direction vector d, the directional derivative is 0.

For a KKT point, if Lagrangian multiplier of a constraint is zero, then the constraint is inactive at this point.

False. For a KKT point, Lagrangian multiplier is zero, the constraint can be either binding or not. For example, $\max x^2$, constraint is $x \geq 0$. Then the KKT condition is $2x + \lambda = 0$,$\lambda x = 0$, $\lambda \geq 0$, $x \geq 0$. Clearly, $(x, \lambda) = (0, 0)$ is a KKT point. And $\lambda = 0$, but the constraint $x \geq 0$ is still active at this point.

2.

Proof.

Use contradiction. Assume that the statement is not true. That it, there is a local minima $\overline{x}$ of function $f(x)$ on S and $\overline{x}$ is not a global minimum.

Since $\overline{x}$ is not a global minimum, there exists $y \in S$, $f(y) < f(\overline{x})$.

Because $\overline{x}$ is a local minimum, there exists $\epsilon > 0$, for any $x \in S$, $|x_i - \overline{x}_i| < \epsilon$, $i = 1, ..., n$, $f(x) \geq f(\overline{x})$.

Let $\lambda = \dfrac{\epsilon}{2d(\overline{x}, y)}$, where $d(\overline{x}, y) = \sum_{i=1}^{n} |\overline{x}_i - y_i|$. Then $\lambda > 0$. And $\lambda < 1$(Otherwise, if $\lambda \geq 1$, $|\overline{x_i} - y_i| \leq d(\overline{x}, y) \leq \dfrac{\epsilon}{2} < \epsilon$, then $f(y) \geq f(\overline{x})$, which is contradict with the assumption $f(y) < f(\overline{x})$.).

Consider the point $z = (1 - \lambda)\overline{x} + \lambda y$.

So $|z_i - \overline{x}_i| = |(1-\lambda)\overline{x}_i + \lambda y_i - \overline{x}_i| = \lambda|\overline{x}_i - y_i| \le \frac{\epsilon}{2d(\overline{x}, y)} d(\overline{x}, y) < \epsilon$, for $i = 1, ..., n$. Then, by the argument above, $f(z) \ge f(\overline{x})$.

However, since f is qusiconvex, by definition, $f(z) = f((1-\lambda)\overline{x} + \lambda y) < \max\{f(\overline{x}), f(y)\} = f(\overline{x})$, which is a contradiction.

So the assumption is not true. $\overline{x}$ is the global minimum.

3.

Solution.

(a). From the function, the gradient is

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} -4x_1 + cx_2 \\ cx_1 - 4x_2 \\ -2x_3 \end{pmatrix}.$$

The Hessian is

$$H(x_1, x_2, x_3) = \begin{pmatrix} -4 & c & 0 \\ c & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The first principle minors are $-4 < 0$, $-4 < 0$, $-2 < 0$;

The second principle minors are $16 - c^2$, $(-4)(-2) = 8 > 0$, $(-4)(-2) = 8 > 0$;

And the third principle minor is $H_3(x_1, x_2, x_3) = -4(16 - c^2)$.

So if $16 - c^2 \ge 0$ and $-4(16 - c^2) \le 0$, then the function is concave. That is, when $c \in [-4, 4]$, the function is concave.

(b). Let the gradient to 0, we have

$$\begin{pmatrix} -4x_1 + cx_2 \\ cx_1 - 4x_2 \\ -2x_3 \end{pmatrix} = 0.$$

If $c = 4$, then all points satisfying $x_1 = x_2$, $x_3 = 0$ are stationary points.

If $c = -4$, then all points satisfying $x_1 = -x_2$, $x_3 = 0$ are stationary points.

If $c \ne 4$ or -4, then $x_1 = x_2 = x_3 = 0$ is a stationary point.

If $c = 4$, $H_2(x_1, x_2, x_3) = 0$ and $H_3(x_1, x_2, x_3) = 0$. The function is $f(x_1, x_2, x_3) = -2x_1^2 + 4x_1x_2 - 2x_2^2 - x_3^2 = -2(x_1 - x_2)^2 - x_3^2 \le 0$. And for all points satisfying $x_1 = x_2$, $x_3 = 0$, the function value is 0. So all points satisfying $x_1 = x_2$, $x_3 = 0$ are global maxima of f.

Similarly, if $c = -4$, $H_2(x_1, x_2, x_3) = 0$ and $H_3(x_1, x_2, x_3) = 0$. The function is $f(x_1, x_2, x_3) = -2x_1^2 - 4x_1x_2 - 2x_2^2 - x_3^2 = -2(x_1 + x_2)^2 - x_3^2 \le 0$. And for all points satisfying $x_1 = -x_2$, $x_3 = 0$, the function value is 0. So all points satisfying $x_1 = -x_2$, $x_3 = 0$ are global maxima of f.

If $-4 < c < 4$, by (a), since $H_1(x_1, x_2, x_3) < 0$, $H_2(x_1, x_2, x_3) > 0$ and $H_3(x_1, x_2, x_3) < 0$, and the function is concave, all stationary points are global maxima, i.e., point $(0, 0, 0)$ is global maxima of f.

If $c > 4$ or $c < -4$, $H_2(x_1, x_2, x_3) < 0$ and $H_3(x_1, x_2, x_3) > 0$, then all stationary points are saddle points, i.e. point $(0, 0, 0)$ is a saddle point.

4.

Solution.

(a). Let $f(x_1, x_2) = 4x_1^2 + x_1x_2 + x_2^2$, $g_1(x_1, x_2) = x_1^2 + x_2^2$, $b_1 = 20$, $g_2(x_1, x_2) = -x_1^2 + x_2$, $b_2 = 0$, and $g_3(x_1, x_2) = -x_2$, $b_3 = 0$.

The KKT condition of P is

$$\begin{array}{ll} \nabla f(x_1, x_2) - \sum\limits_{i=1}^{3} \lambda_i \nabla g_i(x_1, x_2) = 0 & \\ \lambda_i(b_i - g_i(x_1, x_2)) = 0, & i = 1, 2, 3 \\ g_i(x_1, x_2) \leq b_i, & i = 1, 2, 3 \\ \lambda_i \geq 0, & i = 1, 2, 3 \end{array}$$

That is

$$\begin{array}{ll} 8x_1 + x_2 - \lambda_1(2x_1) - \lambda_2(-2x_1) = 0 & \\ x_1 + 2x_2 - \lambda_1(2x_2) - \lambda_2(1) - \lambda_3(-1) = 0 & \\ \lambda_1(20 - x_1^2 - x_2^2) = 0 & \\ \lambda_2(0 + x_1^2 - x_2) = 0 & \\ \lambda_3(0 + x_2) = 0 & \\ x_1^2 + x_2^2 \leq 20 & \\ x_1^2 - x_2 \geq 0 & \\ x_2 \geq 0 & \\ \lambda_i \geq 0, & i = 1, 2, 3 \end{array}$$

(b). First, consider point $(x_1, x_2) = (2, 4)$.

Then

$x_1^2 + x_2^2 = 2^2 + 4^2 = 20$,

$x_1^2 - x_2 = 2^2 - 4 = 0$,

$x_2 = 4 > 0$.

Then $(2, 4)$ is feasible. If this point is a KKT point, then $\lambda_3 = 0$ from the complimentary slackness condition.

So the first two equations in KKT condition are

$20 - 4\lambda_1 + 4\lambda_2 = 0$ and $10 - 8\lambda_1 - \lambda_2 = 0$.

Solve them, we get $\lambda_1 = 5/3$ and $\lambda_2 = -10/3 < 0$.

So point $(2, 4)$ is not a KKT point.

Next, consider point $(x_1, x_2) = (-2, 4)$.

Then

$x_1^2 + x_2^2 = (-2)^2 + 4^2 = 20,$

$x_1^2 - x_2 = (-2)^2 - 4 = 0,$

$x_2 = 4 > 0.$

Then $(-2, 4)$ is feasible. If this point is a KKT point, then $\lambda_3 = 0$ from the complimentary slackness condition.

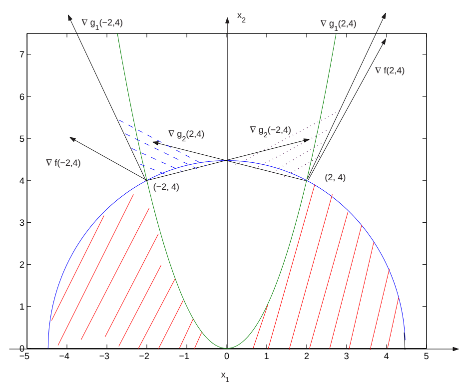
So the first two equations in KKT condition are

$-12 + 4\lambda_1 - 4\lambda_2 = 0$ and $6 - 8\lambda_1 - \lambda_2 = 0$.

Solve them, we get $\lambda_1 = 1$ and $\lambda_2 = -2 < 0$.

So point $(-2, 4)$ is not a KKT point.

(c).



The shaded area by solid lines (red) is the feasible region.

The shaded area by dots (black) is the combination of gradient of active constraints at point $(2, 4)$. Since the gradient of the objective function is not inside the cone, the dual feasibility condition doesn't hold.

Similarly, the shaded area by dashes (blue) is the combination of gradient of active constraints at point $(-2, 4)$. Since the gradient of the objective function is not inside the cone, the dual feasibility condition doesn't hold.

5.

Solution. Let x_i, $i = 1, 2, 3, 4$ be the production quantity of product i.

Then $x_i \geq 0$, $i = 1, 2, 3, 4$.

So the objective function is the profit

$f(x_1, ..., x_4) = \sum_{i=1}^{4}(s_i x_i - pa_i x_i - k_i x_i^2),$

where for each product i, $s_i x_i$ is the revenue, $pa_i x_i$ is the cost on raw materials and $k_i x_i^2$ is the variable cost excluding cost of raw materials.

And we can't buy more than r units of raw materials.

So $\sum_{i=1}^{4} a_i x_i \leq r$.

Hence this is a maximization problem, the formulation is as follows:

$$
\begin{aligned}
\max \quad & f(x_1, ..., x_4) = \sum_{i=1}^{4}(s_i x_i - pa_i x_i - k_i x_i^2) \\
\text{s.t.} \quad & \sum_{i=1}^{4} a_i x_i \leq r \\
& x_i \geq 0, i = 1, 2, 3, 4.
\end{aligned}
$$