Department of Industrial Engineering & Operations Research

IEOR160 Operations Research I

Final Exam **12.20.2004**

Name: **Grade:**

1. **(20 points)** Consider the following optimization problem:

$$
(\mathrm{P}) \quad \begin{array}{lrr}
\max & 3x^2 - 2xy + y^2 - 6y & \\
\text{s.t. :} & -y - 10z \le 0, & x + y + 10z \le 15 \\
& -x + y - 10z \le 0, & -x + 10z \le 7 \\
& x - 10z \le 2, & -x + y + 10z \le 5 \\
& x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \{0,1\}. &
\end{array}
$$

(a) Solve the unconstrained version of (P), i.e., ignore the constraints.

(b) Plot the feasible region of (P) on the $x - y$ space.

(c) Find an optimal solution to (P) and prove that it is optimal.

2. **(20 points)** Consider the following knapsack problem.

$$
\text{(KP)} \qquad \begin{aligned} z = \max\ & 30x_1 + 17x_2 + 14x_3 + 11x_4 + 8x_5 \\ s.t.:\ & 27x_1 + 20x_2 + 16x_3 + 12x_4 + 10x_5 \leq 37 \\ & x_i \in \{0, 1\},\ i = 1, 2, \ldots, 5. \end{aligned}
$$

(a) Provide an *upper bound* on z by solving the linear programming (LP) relaxation of (KP), i.e., the relaxation obtained after dropping the integrality restrictions on the variables. Provide a *lower bound* on z, by rounding the LP relaxation solution.

(b) Solve (KP) by the branch-and-bound method using the following node selection rule: *Always choose the node with the highest LP upper bound for branching.*

3. **(20 points)** Company XYZ needs to meet a demand of d_t units for product P in time periods $t = 1, 2, \ldots, n$. Producing P in time period t costs $\$c_t$ per unit plus a setup charge of $\$a_t$, which is independent of the number of units produced. Company XYZ has the flexibility of producing part of the demand d_t in a period s that is earlier than time period t, in which case it incurs an inventory holding cost of $\$h_i$ for each period $i = s, s+1, \ldots, t-1$ for each unit produced in advance. Company XYZ may also produce part of demand d_t in a period u that is later than time period t, in which case it incurs a penalty $\$p_i$ for each period $i = t, t+1, \ldots, u-1$ for each unit produced late. Company XYZ is interested in finding a production plan that minimizes the total production, inventory, and penalty costs over the following n time periods.

   (a) Show that there exists an optimal solution for XYZ's problem such that demand in any period t is satisfied either entirely from inventory (production from prior periods), or entirely by backlogging (production in future periods), or entirely from production in period t.

   (b) Write down a dynamic programming recursion that can be used to solve the production planning problem of company XYZ. *Write a verbal description of the value function, give a mathematical recursive formula that represents this value function. Indicate how to start and end the calculations.*

4. **(25 points)** A make-to-order manufacturer of a rack and pinion steering for automobiles needs to determine its production plan for model RPS1 for the following n weeks. The order quantity for week t is d_t units. Since the production facility of RPS1 is also used to manufacture other models, the production capacity available for RPS1 is limited to c_t units in week t. Mainly because of this limited capacity, the manufacturer may not meet the demand for RPS1 on time every week. If part of an order (demand for a particular week) is met late, then the manufacturer pays a penalty of a fixed f dollars for that order plus g dollars/unit/week for the delayed units. The cost of production is p dollars per unit and can be assumed to be constant for the planning horizon. In order to avoid excessive penalty costs, especially for orders in peak seasons, the manufacturer is considering to meet part of d_t by subcontracting them to a sister company at a unit cost of v_t dollars. The manufacturer can also meet the demand from inventory; however inventory holding cost is negligible. 5% of the RPS1's produced in-house are defective and 8% of the subcontracted RPS1's are defective. Defective units are scrapped and they have no salvage value. Write down a mathematical programming model with linear objective and linear constraints in order to minimize the total cost of the manufacturer related to the production of model RPS1 over the following n weeks. *Define your decisions variables and state the objective and constraints clearly.*

5. **(15 points)**

(a) Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing function, $g : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.
Define (P1) $\min\{g(x) : x \in S\}$ and (P2) $\min\{h(g(x)) : x \in S\}$.
Prove or disprove that any optimal solution for (P1) is also optimal for (P2).

(b) A vector norm $||.|| : \mathbb{R}^n \to \mathbb{R}$ is a function that satisfies

i) Positivity: $||x|| > 0$ for every $x \neq 0$;

ii) Homogeneity: $|a|||x|| = ||ax||$ for every scalar a;

iii) Triangular inequality: $||x + y|| \leq ||x|| + ||y||$ for every x, y.

Prove or disprove that every vector norm $||.||$ is a convex function.

(c) Let $||.||$ be the Euclidean norm, i.e., $||x|| = \sqrt{\sum_{i=1}^n x_i^2}$. Solve

$$\min \left\{ ||x|| : \sum_{i=1}^{k} x_i = 1; x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \right\},$$ where k is an integer between 1 and n.